DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

A STATISTICAL THEORY OF EQUILIBRIUM IN GAMES

Richard D. McKelvey

Thomas R. Palfrey

SOCIAL SCIENCE WORKING PAPER 955

February 1996

A Statistical Theory of Equilibrium in Games

Richard D. McKelvey   Thomas R. Palfrey

Abstract

This paper describes a statistical model of equilibrium behavior in games, which we call Quantal Response Equilibrium (QRE). The key feature of the equilibrium is that individuals do not always play best responses to the strategies of their opponents, but play better strategies with higher probability than worse strategies. We illustrate several different applications of this approach, and establish a number of theoretical properties of this equilibrium concept. We also demonstrate an equivalence between this equilibrium notion and Bayesian games derived from games of complete information with perturbed payoffs.

JEL classification numbers: 026, 211

Key words: Game Theory, Statistical Methods

A Statistical Theory of Equilibrium in Games*

Richard D. McKelvey Thomas R. Palfrey

1 Introduction

It is well known that the data from experimental games regularly show systematic departures from the most basic predictions of standard game theory. This is true even in the simplest imaginable games, such as two person bimatrix zero sum games with unique mixed strategy equilibrium (O'Neill 1987, Brown and Rosenthal 1990, Ochs 1995). When games have multiple equilibrium, then conventional wisdom typically places further restrictions on the data, such as the restrictions implied by subgame perfection, and the news is even worse (Güth and Tietz 1990, Thaler 1988). The evidence is overwhelming that, at least as a predictive model of behavior, the standard theory of equilibrium in games needs an overhaul. The need of doing this is made even more urgent since we are already regularly apply traditional game theory as a predictive tool.[1]

One possible response is to abandon Nash equilibrium entirely and explain these deviations in terms of psychological theories of behavior. This is not what we are proposing here. While judgement biases, and other sources of deviations from rational behavior surely play some role in these departures, there is still a need to study models of the equilibrium interaction of these effects. Initial attempts at this (McKelvey and Palfrey

*We acknowledge support of National Science Foundation Grant No. SBR-9223701 to the California Institute of Technology and the support of the JPL-Caltech supercomputer project. A version of this paper was presented at the First Japanese Decentralization Conference at Keio University in November, 1994. We are grateful to the warm hospitality during that conference, and appreciate the comments received from the audience. The second author thanks the Laboratoire d'Economie Industrielle and CERAS for research support and hospitality. We acknowledge valuable discussions with Mahmoud El-Gamal, Jacques-François Thisse, and Mark Fey, and the research assistance of Eugene Grayver and Rob Weber.

[1]There are countless examples of this. One obvious one is mechanism design, which has applications to in the design of incentives structures for firms and regulatory agencies, the design of auction and bargaining mechanisms, and public good provision. We also use game theory to make policy prescriptions for government intervention in imperfectly competitive industries, and in fact this has probably been the most significant trend in the field of industrial organization in the last couple of decades. More recently, game theoretic predictions have been applied to a variety of issues that arise in the study of international trade. In fact, if one goes back further, to the 1950's when the first surge of research in game theory began, the bulk of the research was largely motivated (and funded) by applications to arms races and other problems of international conflict and national security.

(1992), El-Gamal, McKelvey, and Palfrey (1993, 1994)) have sought to explain systematic deviations from Nash equilibrium by presuming the existence of specific types of players who do not seek to maximize their expected payoffs in the experimental games. While this approach (and also the application of psychological theories of bias) have had success in explaining some of the departures from Nash equilibrium, they are both open to the criticism of being post hoc, in the sense that the "invention" of types is specific to the game under consideration. What would be more desirable is a general model that does not need to be specially modified for each game and each dataset.

This paper presents one such class of general models. We consider a very simple extension of the standard model of Nash equilibrium which can explain some of systematic violations of Nash equilibrium. The basic idea is that there are many different possible "types" of players, and that the statistical variation in behavior that is always observed in experiments is due to some underlying distribution of these types. As a result, behavior will be stochastic, even if all agents have private information about their type and always adopt pure strategies. More importantly, because of the game setting, players will respond to anticipated errors of opponents in ways which can lead to systematic deviations from Nash equilibrium.

We introduce this idea first in its "reduced form" which supposes that individual behavior is not perfectly optimal. Specifically, given a set of alternative choices, individuals choose probabilistically, choosing better alternatives more often than worse alternatives. The Nash equilibrium model thus corresponds to an extreme special case of this model, in which the probability of choosing an optimal alternative is equal to one. Thus our model specifies a choice probability function that depends on the vector of expected utilities of all available choices.

In a game, the available choices to a player are given by his or her strategy set. The expected utility of each strategy for a player is determined by the probability distribution of the strategies of the other players of the game. Thus, the choice probability function in our model is a generalization of the "Best Response Correspondence" in game theory. We simply call it a **Quantal Response Function**, and assume it is continuous and positively responsive to the expected payoff of each strategy. A *Quantal Response Equilibrium* is a fixed point of the quantal response functions, just as a Nash equilibrium is a fixed point of the best response correspondence.

We then narrow our focus to a particular class of response functions associated with the logit model of choice. Each individual is assumed to have the same logit response parameter (which indexes how close it is to the standard best response model). We term the quantal response equilibria of this parametric model, **Logit Equilibria**. Given a particular game, we then define the logit equilibrium correspondence for that game as a function of the logit parameter. McKelvey and Palfrey (1995a) establishes a number of technical properties of the logit equilibrium correspondence. We summarize them here and illustrate these properties with a series of examples.

There are a number of models of subrational behavior in games that are related to quantal response equilibrium. Rosenthal (1989) and Chen, Friedman, and Thisse (1995) explore models which also look at fixed points of statistical response functions. Van Damme (1987) presents a model in which it is costly for players to adopt pure strategies. This "control cost" model generates statistical response functions. Schmidt (1992) investigates games of reputation building in which players choose an optimal strategy with probability ε and a suboptimal strategy with probability $1-\varepsilon$, solves for the Bayesian Nash equilibrium of the resulting game, when this is common knowledge among the players, and applies this to the analysis of experimental chain-store-paradox games. Zauner (1993) models behavior in the centipede game by introducing Normally distributed payoff disturbances to the agent form of the game and then computes the resulting Bayesian equilibrium. This is essentially the same as looking at a quantal response equilibrium with "Probit" response curves.[2] Stahl and Wilson (1995) apply a version of the logit equilibrium model to analyze data from a series of experimental two-person games they conducted. Beja (1992) proposes a model of imperfectly rational play in games, where players are limited in their ability to exactly implement their strategy choices. Ma and Manove (1993) apply a similar idea to study bargaining games when the players' cannot perfectly implement their bargaining strategies.

This paper is organized as follows. In the next section, we lay out the formal structure of the model for normal form games. In sections 2 and 3, we discuss the theoretical foundations of quantal response equilibrium, and show how it can be interpreted as a Bayesian equilibrium with Harsanyi-type payoff disturbances. Section 4 presents a parametric family of quantal response models, based on logit response curves. We call this the logit equilibrium. Section 5 shows how the logit equilibrium correspondence implies refinements of Nash equilibrium. We then apply this refinement to a simple bargaining game called "The Ultimatum Game." Section 6 discusses a number of extensions of the basic model: infinite strategy spaces, games of incomplete information, heterogeneity, endogenizing the response curves, and games in extensive form. We make some concluding remarks in section 7.

2 The Model

We begin by defining a game in its normal form in the standard way. Let $I = \{1, \ldots, n\}$ be the set of **players**. For each $i \in I$ there is a strategy set A_i, which we assume to be finite, with J_i elements. Each player has a payoff function $u_i : A \to \mathcal{R}$ where $A = \prod_{i \in I} A_i$ Let S_i be the set of probability distributions over A_i and an element $s_i \in S_i$ is a **mixed strategy**. Given a strategy profile $s \in S = \prod_{i \in I} S_i$ player i's expected payoff is $v_i(s) = \sum_{a \in A} p(a) u_i(a)$, where $p(a) = \prod_{i \in I} s_i(a_i)$. A (mixed) strategy profile $s \in S$ is a **Nash Equilibrium** if, for all $i \in I$ and for all $t_i \in S_i$, $v_i(s) \geq v_i(t_i, s_{-i})$.

We next define a response function R_i, which, for a given player i, maps a vector of

[2]For an analysis of centipede games with logit response curves, see Fey, McKelvey, and Palfrey (forthcoming) and McKelvey and Palfrey (1995b).

payoffs for each possible pure strategy into a mixed strategy for i. That is, $R_i : \mathcal{R}^{J_i} \to S_i$. We denote the probability assigned to strategy a_{ij} by R_{ij}. A special case of a response function is the *Best Response Function*, which is multivalued and upper hemicontinuous, but not continuous. Here we wish to restrict attention to response functions that are *single-valued* and *continuous.*

Let $v_i = (v_{i1}, \ldots, v_{ij}, \ldots, v_{iJ_i})$ be a vector of expected payoffs, one for each possible pure strategy in S_i. We make the further two assumptions that:

1. R_i is increasing in each argument. That is, suppose that v_i and v_i' differ only in the j^{th} component. Then $R_{ij}(v_i') \geq R_{ij}(v_i)$ if and only if $v_{ij}' \geq v_{ij}$.

2. R_i assigns higher probabilities to better strategies. That is R_i preserves the order of v_i, so that $R_{ij}(v_i) \geq R_{ik}(v_i)$ if and only if $v_{ij} \geq v_{ik}$.

These assumption guarantee that the response functions are positively responsive. There are many different possible response functions that satisfy these restrictions.

Given a set of response functions, $R = R_1, \ldots, R_I$, one for each player, and given a normal form game, (I, S, u), one can define an equilibrium relative to R as a fixed point of the following mapping from S to S. For any given $s \in S$, $R_i(v_i(s))$ maps s_i into a new element of S_i. Since u is continuous in s, R is a continuous function mapping a compact convex set into itself, and therefore has at least one fixed point. We call any such fixed point a R-**response equilibrium** of the game (I, S, u).

3 Foundations of R-Response Equilibria

The basic idea that choice probabilities are related in a continuous and monotonic way to expected payoffs has been around a long time. It was first formalized in the mathematical psychology literature (Thurstone 1927, Luce 1959, Luce and Suppes 1965) where it was applied to the study of data from individual choice experiments. The new dimension in applying the model to game theory is that the choice probabilities of one player affect the expected utilities of the other players' choice alternatives, so that the choice probabilities themselves are endogenously determined. That is, random individual behavior has equilibrium effects.

There are a number of ways to think about the adoption by players of responses that are not necessarily "best" responses. Two immediately come to mind. First one could interpret this as a departure from rationality by the players. In this sense, the model falls into the class of "bounded rationality" models. This is the interpretation given in Rosenthal (1989) and Chen, Friedman, and Thisse (1995), and, as they point out, is the basic motivation behind the large literature in mathematical psychology called "probabilistic choice." The idea here is that behavior is subject to choice errors (i.e., nonoptimal choice) by individuals, but these choice errors have a specific relationship to the true expected payoffs: "big" mistakes (i.e., involving a large loss of utility relative to the optimal choice) are made less often than little mistakes.

An alternative interpretation of smoothed best responses is consistent with standard game theory, following results by Harsanyi (1973) about Bayesian equilibria of games of incomplete information. The basic idea there is to think of the game (I, S, u) as a complete-information approximation of a more complicated game of incomplete information, in which the actual payoffs of the players are private information. These actual payoffs are equal to the payoffs given by (I, S, u), but include additive disturbances from a known distribution. The disturbances to player i's payoffs are known only to i. This generates a game of incomplete information with player types determined by the individual-specific payoff disturbances.

This second interpretation is the basis for the model developed by McKelvey and Palfrey (1995a), and is also related to the random-utility interpretation of randomness in discrete choice developed in the econometrics literature (see McFadden, 1976.) The idea here is to interpret the random disturbances as latent variables which cannot be observed by an outsider (say an econometrician or another player in the game). However, each player is assumed to observe the sum of this latent variable and the true payoff. In this sense, one can think of the perturbed payoffs as simply being each player's estimate of the expected utility of a particular strategy, with the latent variable representing the estimation error. This is formalized below.

For each i and each $j \in \{1, \ldots, J_i\}$, and for any $s \in S$, denote by $v_{ij}(s)$ the expected utility to i of adopting the pure strategy a_{ij} when the other players use s_{-i}. For each pure strategy a_{ij}, player i does not receive payoff $v_{ij}(s)$ but instead receives:

$$\hat{v}_{ij} = v_{ij}(s) + \varepsilon_{ij}.$$

Player i's profile of payoff disturbances, $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iJ_i})$, is distributed according to a joint distribution with density function $f_i(\varepsilon_i)$. Assume that the marginal distribution of f_i exists for each ε_{ij} and $E(\varepsilon_i) = 0$. McKelvey and Palfrey (1995a) call $f = (f_1, \cdots, f_n)$ **admissible** if f_i satisfies the above properties for all i. The assumed choice behavior is that each player chooses strategy a_{ij} such that $v_{ij} \geq v_{ik}\ \forall k = 1, \ldots, J_i$. Given this choice behavior, v and f induces a distribution over the actual choices by each player. To be more specific, for any v, define $B_{ij(v)}$ as the set of realizations of ε_i such that strategy a_{ij} has the highest estimated (or disturbed) expected payoff. So

$$P_{ij}(v) = \int_{B_{ij}(v)} f(\varepsilon)d\varepsilon$$

is the induced probability that player i will select strategy j given v. Since $P(v) \in S$ and $v = v(s)$ is defined for any $s \in S$, $P \circ v(s) = P(v(s))$ defines a mapping from S

into itself. Any fixed point s^* such that $s^* = P(v(s^*))$ is called a **Quantal Response Equilibrium** of the game (I, A, u).

The following results are immediate, and the proofs are left to the reader:

- For any admissible f, the mapping $P \circ v$ a continuous response function
- For any admissible f, the mapping $P \circ v$ has a fixed point
- Given f, a quantal response equilibrium is a R-Response Equilibrium, where $R = P$

4 Specific parametric models of R-Response Equilibrium

Here we describe three different models of response equilibrium, which differ in their assumptions about the shape of the R function that maps the vector of expected payoffs to an agent into a vector of choice probabilities. The first was proposed by Rosenthal (1989) and is simply a linear response model which applies to games in which players have only two strategies. The second two are derived from the logit model of individual choice.

4.1 Linear Response Model

Rosenthal's (1989) T-equilibrium was developed as a model of boundedly rational behavior in games where players have only two strategies. That model specifies that the probability an individual adopts a particular strategy is a linearly increasing function of the difference between the expected payoff of that strategy and the expected payoff of the other strategy. If the two strategies yield the same expected payoff (given the choice probabilities of the other players) then each is used with probability $\frac{1}{2}$. More generally, if $P_{i1} = \min\{\max 0, .5 + B(v_{i1} - v_{i2}), 1\}$ and $P_{i2} = 1 - P_{i1}$, where B is a nonnegative constant representing the degree of rationality of the players. As B approaches infinity, the linear response model approaches the Nash equilibrium.

4.2 Smooth Response Models

There are a number of possible examples of models using R functions that are smooth. One is the Luce model (Luce 1959), in which the odds of using one strategy as opposed to another strategy are proportional to the ratio of the expected utility of the strategies. This, and a generalization of it, where the odds are given by the ratio of expected utilities taken to some (nonnegative) power, μ, is examined by Chen, Friedman, and Thisse (1995).

An alternative model (see McKelvey and Palfrey, 1995a, for a technical analysis of its properties) is the logit model where the odds are determined by the exponential function of the utilities times a given nonnegative constant, λ. Specifically:

$$P_{ij}(v_i) = \frac{e^{\lambda v_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda v_{ik}}}$$

This response function arises from the assumption that the distribution $f_i(\varepsilon_i)$ of payoff errors follows a type I extreme value distribution, with cumulative density function $F(\varepsilon_{ij}) = e^{-e^{-\lambda\varepsilon_{ij}}}$, where the $varepsilon_{ij}$ are independent. This distribution of errors is assumed in much of the discrete choice econometrics literature (see McFadden, 1973, 1976). Here, the parameter λ plays the same role as B in the linear model. As λ approaches infinity, the model of behavior approaches the best response model. When $\lambda = 0$ behavior is essentially random, as all strategies are played with equal probability. The quantal response equilibrium in this model is given by a vector of choice probabilities, one for each player, $(s_1, \ldots, s_i, \ldots, s_I)$ such that, for each $i \in I$ and for each $a_{ij} \in A_I$

$$s_i \cdot = \frac{e^{\lambda x_{ij}}}{\sum_{k=1}^{J_i} e^{\lambda x_{ik}}}$$

where $x_{ij} = v_{ij}(s)$.

This produces a system of $J = \{\sum_{i=1}^{I} J_i\}$ equations, and it is easy to prove that this system always has a solution for any value $\lambda \in [0, \infty)$ ((McKelvey and Palfrey 1995a). Any such point is called a **logit equilibrium** for λ. The set of all (λ, s) such that s solves the above equation is called the **Logistic Equilibrium Correspondence**. Moreover, any limit point of a sequence of **Logit Equilibria** as λ approaches infinity is a Nash equilibrium of the underlying game.

5 Properties of logit equilibria

Logit equilibria have a number of properties that are different than Nash equilibrium predictions.

5.1 Use of dominated strategies

A Nash equilibrium cannot use a strongly dominated strategy with positive probability. Various equilibrium refinements typically eliminate any Nash equilibria that use weakly dominated strategies. In contrast, a logit equilibrium uses every strategy with positive probability. The probability of adopting a strongly dominated strategy goes to zero as λ goes to infinity. However weakly dominated strategies can be used with positive probability even in the limit, as is shown in examples later in this paper.

5.2 Mixed strategy equilibria

It is a property of mixed strategy Nash equilibria that the mixing probabilities do not depend on the players *own* payoffs over outcomes in the support of the equilibrium, but only on the *other players'* payoffs. Since a player is by definition indifferent between all strategies in the support of the mixed equilibrium, the only reason for mixing correctly is to keep the other players at the Nash equilibrium. In the logit equilibrium, in contrast, the mixing probabilities depend on the players' own payoffs. To see this, consider the game of Table 1, where $x > 0$. Let p represent the probability the row player chooses "U", and q represent the probability that the column player chooses "L". This game has a unique Nash equilibrium at $p = q = \frac{1}{2}$ for all x.

	L	R
U	x,0	0,1
D	0,1	1,0

Table 1: A game with a unique mixed strategy equilibrium.

To see what the logit equilibrium looks like, set $r = \frac{p}{1-p}$ and $s = \frac{q}{1-q}$. The equations for a logit equilibrium can then be written as

$$ln(r) = \frac{\lambda(sx-1)}{s+1},$$

and

$$ln(s) = \frac{\lambda(1-r)}{1+r}.$$

When $x = 1$, $r = s = 1$ (i. e., $p = q = \frac{1}{2}$) is a solution to the above equations for *all* values of λ. To see what happens when $x > 1$, solve for s in the first equation yielding

$$s = \frac{\lambda + ln(r)}{\lambda x - ln(r)}$$

(Note for $0 < \lambda < \infty$ the denominator and the left hand side must be positive. So the numerator must also be positive). Now, substituting into the second equation, and then differentiating, we get that the derivative r' of r with respect to x is

$$r'(\frac{x+1}{r(\lambda + ln(r)} + \frac{2(\lambda x - ln(r))}{(1+r)^2}) = 1.$$

Since all terms in the fractions are positive whenever $x \geq 0$ and $\lambda > 0$, it follows that r' (and hence $p' = \frac{r'}{(1-r)^2}$) is positive. Thus, for the logit equilibrium, the mixing probability *is* a function of the player's own payoff. For all values of $0 < \lambda < \infty$, the probability of player 1 choosing "U" increases as x increases.

5.3 Payoff magnitude effects

Since any game theoretic analysis starts from the assumption that utilities are expressed as Von Neumann-Morgenstern utility functions, it is a property of game theory, and hence of any Nash equilibrium that predictions are unaffected by changes in the magnitude and scale of the utility function. Thus, any game theoretic analysis seems unable to account for payoff magnitude effects.

Since the logit equilibrium is a Bayesian equilibrium, the same conclusion applies here, as long as the *same* transformation is applied to the payoffs of the game and the payoff due to the error terms. However, if the payoff from the error arises from considerations outside of the game, then a change in magnitude of the game payoffs would affect the utilities in the game, but *not* the utilities contributing to the error term. In this situation, the prediction of the logit equilibrium *would* change.

To see how the logit equilibrium predictions would change, note that in the logistic equation for $P_{ij}(v_i)$, λ and v_{ij} always enter as scalar multiples of each other. Thus, multiplying all utilities by a constant is equivalent to multiplying λ by a constant. Thus, if the precision parameter of the error term is λ, and all of the game utilities (for all players) are multiplied by a factor of a, then the resulting logit equilibrium would be the same as a logit equilibrium for the original game under the assumption that the precision is $a\lambda$. It follows from results to be shown below, that in general this would imply that higher payoffs would lead to outcomes "closer" to Nash behavior. The case when the magnitudes for one player are changed, and the other player are not is not so clear cut. Here, multiplying one player's utilities by a constant while the other players were unchanged would be equivalent to introducing an individualized precision λ_i for that player, where $\lambda_i = a\lambda$. We discuss these effects again later in the context of heterogeneity of errors.

6 Refinements of Nash equilibrium implied by the logit equilibrium

Following Harsanyi (1973), call a Nash equilibrium, σ^*, **approachable** if there exists a sequence of λ's approaching infinity and a corresponding sequence of logit equilibria that converges to σ^*. It is easy to show that this set is non-empty, since the logit equilibrium correspondence is upper hemicontinuous. The set of approachable equilibria thus defines a refinement of Nash equilibrium. A final property of logit equilibria is that it defines (for almost all games) a unique selection from the set of Nash equilibrium. For small enough values of λ, it can be shown that there is a unique logit equilibrium close to the "centroid" of the game. This is the point where all strategies are adopted with equal probability. We define the **principal branch** of the logit equilibrium correspondence to be the branch which starts at the centroid. For generic games, the principal branch is a one dimensional manifold, and hence can be followed for values of λ ranging from 0 to infinity.[3] The limit point of the principal branch as λ approaches infinity is called the **logit solution** of the game and is denoted σ_L^*. Thus the logit equilibrium correspondence can be used to define one weak refinement (approachability) and one strong refinement (the logit solution).

We illustrate these ideas in a few simple examples below.[4]

Example: The asymmetric game of chicken

The first example is an asymmetric version of the game of chicken. If both players play "soft" then they each get an intermediate payoff. If they both play tough they both get very low payoffs. If they choose different strategies, then the tough player receives a very large payoff, and the soft player receives a low payoff. The payoffs are given below, in Table 2.

	tough	soft
tough	0,0	6,1
soft	1,14	2,2

Table 2: An asymmetric game of Chicken

In this game there are three Nash equilibria, two corresponding to pure strategy equilibria where one player is tough and the other is soft, and the third corresponding to both players mixing. Usual refinements provide no help in selecting among equilibria. Figure 1 displays the logit equilibrium correspondence for this game. As one can see, the weak refinement of approachability by a sequence of logit equilibria does not provide any help, as all three equilibria are approached by logit equilibria. The unique component of the equilibrium graph converges to the pure strategy equilibrium where the row player plays soft and the column player plays tough. This is the equilibrium favoring the player (column) who benefits the most from playing tough. In fact this would seem to be a general property[5] of the class of asymmetric chicken games where the only asymmetry is in the payoff to the tough player when one player is tough and the other is soft. The logit solution always picks out the pure strategy equilibrium favoring the player benefitting the most from being tough.

Example: $J \times J$ Ultimatum Games

The second example is a very simple version of a bargaining game that has become known as "The Ultimatum Game" (see Güth and Tietz 1988 and Thaler 1988). It has been

[3]The basic idea and the mathematical tools needed to prove this are similar to the "tracing procedure" of Harsanyi and Selten (1988). But the selection that is made by the two solutions is different.

[4]The interested reader should also see the papers by Chen, Friedman, and Thisse (1995) and McKelvey and Palfrey (1995a), which give a number of other examples.

[5]We do not have a formal proof of this, but all the solutions we have computed have this property. A similar property appears to hold for asymmetric Battle of the Sexes games.

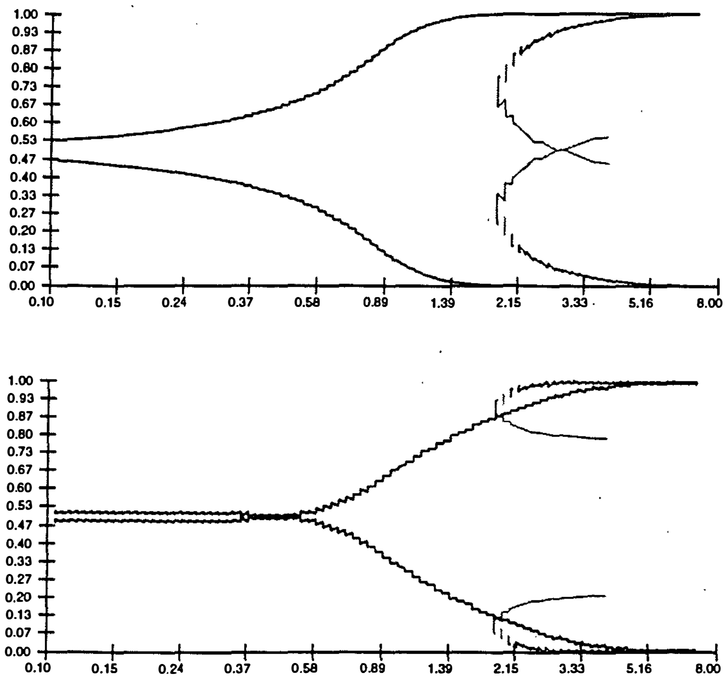


Figure 1: QRE Correspondence for game of Chicken. Probability of "tough" for row player (top graph) and column player (bottom graph)

the subject of a great deal of experimental work by both economists and psychologists, largely because the results strongly contradict the predictions of subgame perfect Nash equilibrium. In this game, there are two players, who are given the task of dividing a fixed sum of money according to the following simple rules. Player 1 offers a split of the money, and then Player 2 either accepts or rejects the proposed division. If the offer is rejected by Player 2, the game ends and neither player receives anything. Assume for the purposes of analysis that there are a finite number of feasible splits[6] (say, in one dollar increments). Then subgame perfect Nash equilibrium implies that Player 2 will receive, in equilibrium, either 0 or the lowest possible positive amount.

A simplified version of the ultimatum game is easy to study in the context of the logit model. In this simplified version of the game[7], the pie consists of $2J$ dollars and there are J permissible splits of the pie, ranging from the 50/50 split (where each player receives J dollars) and the most favorable split for player 1, subject to the constraint that player 2 receives a positive amount. (I.e., Player 1 receives $2J-1$ dollars and Player 2 receives 1 dollar.) The second player writes down a minimum acceptable offer (some number between 1 and J). If Player 1 offers at least this minimum level, then the outcome is exactly the proposed split. If not, then both players receive 0. The normal form of this game is given below in Table 3.

[6]In all experiments, there are a finite number of choices available for Player 1 to offer Player 2

[7]Gale, Binmore, and Samuelson (1993) investgate the evolutionary dynamics of the J=2 version of the game.

	3	2	1
3	3,3	3,3	3,3
2	0,0	4,2	4,2
1	0,0	0,0	5,1

Table 3: The 3×3 simplified ultimatum game.

It is well known that this game has multiple Nash equilibria (in fact a continuum of Nash equilibria), only one of which is trembling hand perfect. In the perfect equilibrium, Player 1 offers $(5, 1)$ and Player 2 chooses the third strategy (accept anything). Figure 2 shows the unique continuous selection from the logit equilibrium correspondence for values of λ ranging from 0 to 10. This shows clearly that the logit solution to this game is *not* the unique perfect equilibrium, but rather results in the $(4, 2)$ split. Player 2 does not adopt a pure strategy, but rather mixes evenly between the choices of "accept anything" and "reject only the worst offer." In addition to the limiting properties of this logit equilibrium mapping, it is also instructive to look at the solution for low and intermediate values of λ. At low values of λ both players are nearly random, so the optimal strategy for Player 1 is actually the 50/50 split. Accordingly, for sufficiently small λ this is the modal offer. The probability of using the strategy of making the greediest offer declines very quickly in λ. For higher values of λ, the modal offer becomes $(4, 2)$. For Player 2, the strategy of accepting only the equal split disappears very quickly.

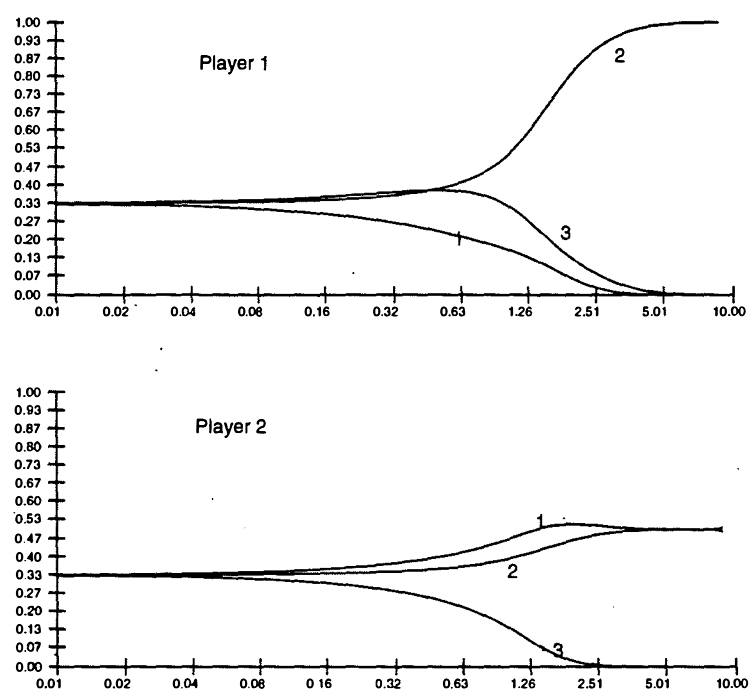


Figure 2: Logistic QRE correspondence for the 3x3 ultimatum game

Of course with only three strategies for each player it is difficult to tell whether or not the modal strategy in the logit solution will be close to the perfect equilibrium. To extend this example, consider the 6×6 version of the game, illustrated in Table 4 below.

	6	5	4	3	2	1
6	6,6	6,6	6,6	6,6	6,6	6,6
5	0,0	7,5	7,5	7,5	7,5	7,5
4	0,0	0,0	8,4	8,4	8,4	8,4
3	0,0	0,0	0,0	9,3	9,3	9,3
2	0,0	0,0	0,0	0,0	10,2	10,2
1	0,0	0,0	0,0	0,0	0,0	11,1

Table 4: The 6×6 simplified ultimatum game.

Figure 3 shows the unique connected component of the logit equilibrium graph. The logit solution is now the $(9,3)$ split. As in the 3×3 version of the game, the modal offer begins with the equal split, and as λ increases, the modal offer progressively increases until it finally settles at the $(9,3)$ split. An interesting theoretical question would be what the logit solution for the equilibrium split converges to (as a fraction of $2J$), as J goes to infinity.

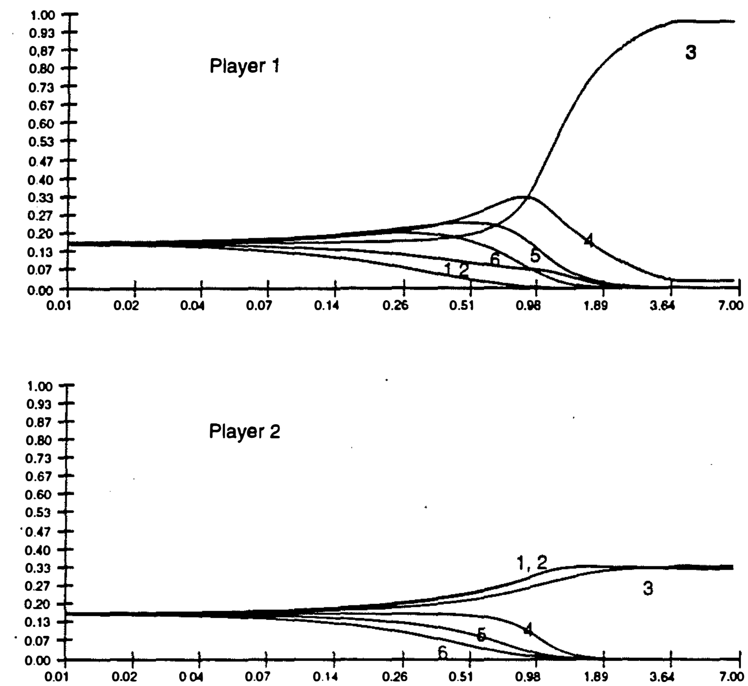


Figure 3: Logistic QRE for the 6x6 ultimatum game

7 Extensions

Many applications to economics involve games with infinite strategy spaces, private information, and sequential play. This includes, for example, voluntary contribution games in the provision of public goods, auctions, bargaining, principal agent models, oligopoly, spatial competition, and so forth. Here we explain how the logit model can be extended to accommodate these different applications. Other parametric models of smooth response functions can be extended in a similar fashion

7.1 Infinite Strategy Spaces

If strategy spaces are infinite[8], then the probability of adopting any particular pure strategy will be zero. However, it is possible to retain the basic idea that the odds of one strategy vs. another strategy are given by the ratio of an exponential function of the expected utility of each strategy. That is, we still want it to be the case that, for any $a, a' \in A_i$, the ratio, is given by:

$$p(a)/p(a') = \frac{e^{\lambda v_a}}{e^{\lambda v_{a'}}}$$

Thus, $p(a)$ is given by:

$$p(a) = \frac{e^{\lambda v_a}}{\int_{A_i} e^{\lambda v_t} dt}$$

as long as $\int_{A_i} e^{\lambda v_t} dt$ is well-defined. In general, if A_i is infinite, then it may or may not be possible to define a probability distribution function (density function) $p(a)$ satisfying this logit condition for all $a, a' \in A_i$. It depends on whether or not the integral of $e^{\lambda v_a}$ with respect to the Lebeague measure is guaranteed to be finite. We restrict $p(a)$ to be a measurable function. Further assumptions are needed on the game to ensure that the logit response function is well-behaved.

For example, if A_i is a compact subset of $\mathcal{R}^n$ and the underlying payoff function of the game is continuous, then $\int_{A_i} e^{\lambda v_t} dt$ is well-defined. There are some additional technical issues that must also be addressed, as discussed in some of the literature on the econometrics of probabilistic choice models.

7.2 Games of Incomplete Information

If players have private information, represented by **Type Sets**, $(T_1, \ldots, T_I)$, and type contingent beliefs, $\pi_i(t_i)$ which specify Player i's beliefs about the types of the other players. Then the extension of strategies, is a **Type Contingent (Mixed) Strategy**, which, for each player, maps T_i into S_i. Such a strategy will be denoted σ_i and, for each $t \in T_i$ and for each $a_{ij} \in A_i$, $\sigma_{ij}(t)$ denotes the probability player i plays strategy a_{ij} under σ_i. Payoffs may depend on the entire profile of types, as well as the profile of actions. The

[8] Some natural applications include oligopoly and spatial competition. Some of these are studied in the context of a generalized version of Luce's random choice model in Friedman, Thisse, and Palfrey (1995).

extension of Nash equilibrium to this kind of game is called a Bayesian Nash equilibrium, and a similar extension is possible for the logit equilibrium (or, more generally, quantal response equilibrium). Given a strategy profile σ, this implies an *effective mixed strategy* s_i for each player, which is a probability distribution over A_i, where the probability of strategy a_{ij} just given by

$$s_{ij} = \int_{T_i} \sigma_{ij}(t)dF(t).$$

If the type space is finite, then the definition is the same, except the integral is replaced by a finite sum over the types in T_i. Given an effective mixed strategy profile, s, the probability each type of each player assigns to each strategy is given by the logit equation below:

$$\sigma_{ij}(v_i(s)) = \frac{e^{\lambda v_{ij}(s)}}{\sum_{k=1}^{J_i} e^{\lambda v_{ik}(s)}}$$

Where $v_i(s)$ is defined as before. Then, given the parameter, λ, a logit equilibrium is any σ satisfying the above system of simultaneous logit equations where s is the effective mixed strategy under σ. It is straightforward to show in the case of finite T that a logit equilibrium exists for any game and for any nonnegative value of λ, and that the logit equilibrium correspondence has the same properties as in the model for games with complete information. If T is a compact rectangle in R^n then we also have existence, under appropriate regularity conditions on the joint distribution of types. We illustrate this equilibrium for the following public goods game, which has been studied extensively in experiments by Palfrey and Rosenthal (1992, 1993, 1994).[9]

Example: Voluntary provision of a binary public good.

There are I players, each of whom is endowed with a single indivisible unit of private good. Each player independently decides whether to "contribute" their unit of private good toward the production of the public good. If at least $K \leq I$ units are contributed, then the public good is provided. (Unused contributions are not rebated.) Each player values the public good the same (which we normalize at 1). Each player's valuation of their discrete unit of the private good is private information, and this value is denoted by c_i, which we call i's **contribution cost**. These valuations are independently drawn from a commonly known probability distribution denoted $F(c)$. The payoff function to player i is thus given as follows:

$u_i = 1 + c$ if i does not contribute and at least K others contribute

[9] See also the theoretical analysis in Fudenberg and Tirole (1991).

$u_i = 1$ if i contributes and at least $K-1$ others contribute

$u_i = c$ if i does not contribute and fewer than K others contribute

$u_i = 0$ if i contributes and fewer than $K-1$ others contribute

For this example, assume that the distribution of costs is uniform over the interval $[0, 1.5]$, $N = 3$ and $K = 2$. A symmetric Bayesian equilibrium is represented by a cutpoint strategy, c^*, according to which each individual contributes if and only if their contribution cost is less than or equal to c^*. It is easy to see that such a cutpoint must equal the probability that an individual's contribution is critical to the production of the public good, given that the other players are adopting the same cutpoint decision rule.[10] For this example, the pivotal condition reduces to:

$$c^* = 2c^*(1-c^*)/2.25$$

There are two Bayesian equilibria of this game, one in which no one ever contributes, i.e., $c^* = 0$ (which can be shown to be unstable) and another in which $c^* = .375$, which produces an effective mixed strategy of each player contributing with probability .25.

The logit equilibrium is not a cutpoint equilibrium since, for every contribution cost, and for every value of λ, the probability of contribution must lie strictly between 0 and 1. Fixing λ, this defines a function assigning a probability of contribution to each value of c. Call this function $P(c)$. Then define:

$$Q = \int_c P(t)f(t)dt$$

and

$$P(c) = \frac{e^{\lambda(2Q(1-Q)-c)}}{1+e^{\lambda(2Q(1-Q)-c)}}$$

While we do not have an analytical solution, it is fairly easy to obtain numerical solutions, for any value of λ. The numerical solutions for Q, as a function of λ, are graphed in the darker curve displayed in Figure 4 below.

Notice that the probability of contribution is greater than the Bayesian equilibrium (.25) for all values of λ. This is interesting because it is consistent with similar experimental findings (Palfrey and Rosenthal 1991, 1992, 1993) where contribution levels in

[10] In some cases, but not in this example, an equilibrium of always contributing regardless of one's contribution cost may exist, in which case, the probability of being pivotal must be greater than or equal to the upper bound on the distribution of costs. This could happen for example if $N = K$ and the maximum value of c were less than 1.

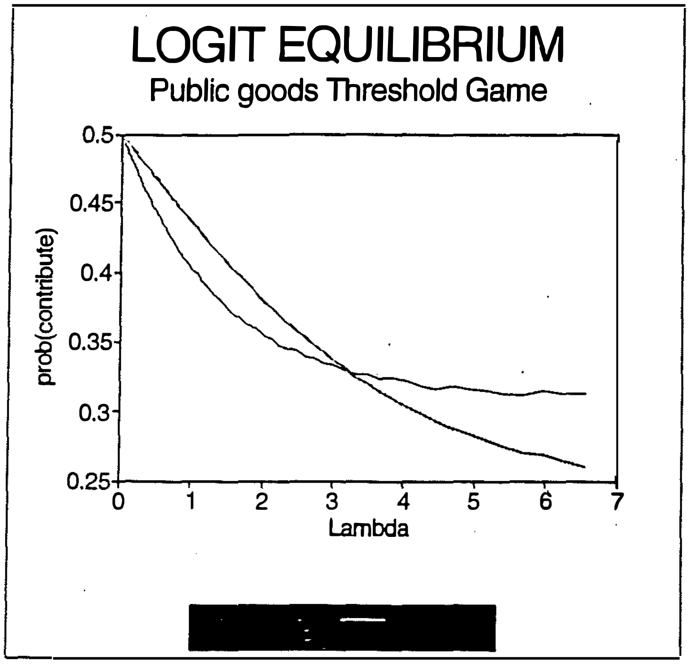


Figure 4: Q as a function of λ for voluntary contribution games

this game were systematically greater than the Bayesian equilibrium contribution levels. They reported that extent of overcontribution varied considerably depending on the exact values of K, N, and the distribution of costs. For example, less overcontribution is observed in a similar game in which K is reduced from 2 to 1. The logit equilibrium graph of this variation is shown as the lighter curve in Figure 4. Notice that it converges much more rapidly as a function of λ for this variation of the game, suggesting a possible theoretical explanation for this apparent anomaly.

7.3 Heterogeneity

The models described above, including the extensions, assume that all players have the same parameter, λ.[11] However, there are good reasons to believe that many situations correspond to environments in which different players' response functions are different. The simplest way to model this is to assume that different players have different values of λ. As an example, consider a chess game between an expert and a beginner. If this were common knowledge, then the expert might adopt a different strategy than she would against another expert. The same might be true of two bargainers, one who is very experienced at negotiation and the other who is not. Such an example might be the haggling that often occurs at flea markets between the sellers (who are typically very experienced at haggling) and the buyers (many of whom frequent flea markets only occasionally). As a third example, if one were to apply this model to entry deterrence by

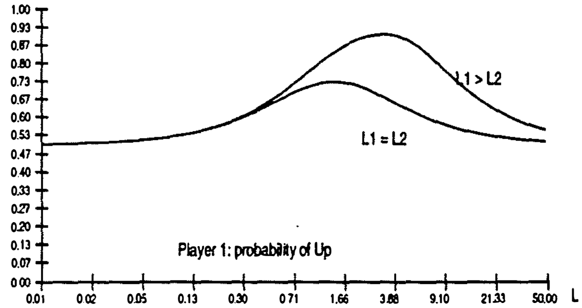
[11]Implicitly it is also assumed that this is common knowledge, at least in the Harsanyi interpretation of the logit equilibrium as a Bayesian equilibrium of a game of incomplete information.

a monopolist, one might wish to assume that the entrant has a lower value of λ than the incumbent. Finally, some experiments have been conducted using subjects of different levels of experience, and it would be reasonable to assume that the more experienced players have higher λ's.

Example: Illustration of heterogeneity

Consider the game of Table 1, setting $x = 4$. This is similar to an experimental game reported in Ochs (1995).

In Figure 5, we graph the logit equilibria of this game, first under the assumption of homogeneous λ ($\lambda_1 = \lambda_2 = \lambda$) and second under the assumption that the λ for Player 2 is one fourth the value of the λ for Player 1 ($\lambda_1 = 4\lambda_2 = \lambda$).[12]



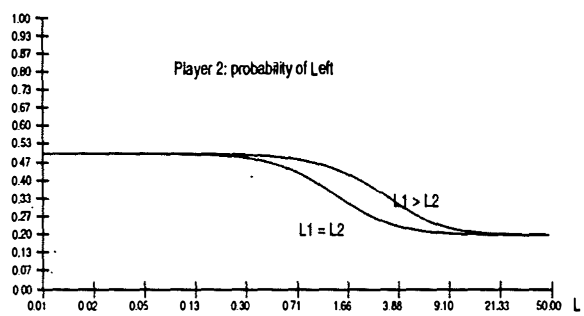


Figure 5: The effects of heterogeneity

We see from the figure that the decreased accuracy of player 2 results in that player staying closer to random behavior and further from the Nash equilibrium for all values of λ. This also results in an adjustment to player 1's strategy. Since player 1 expects player 2 to be less accurate, he can now take advantage of the fact that player 2 is playing L too frquently, and adopt U with higher probability than the Nash prediction of $\frac{1}{2}$.

In principle, one could extend the model of heterogeneity even further, by dropping the assumption that the profile of λ's is common knowledge among the players. One

[12]Notice that this is equivalent to the logit equilibria in which all payoff of Player 2 are multiplied by a factor of $\frac{1}{4}$.

alternative would be to assume that each player's λ is drawn from a commonly known distribution.

7.4 Endogenizing λ

The question of heterogeneity leads a step further to the more general issue of what determines the level of λ. There are at least two possibilities which are both worth further consideration, but are beyond the scope of this paper. The first is that, if one interprets λ as a measure of how *carefully* a player is making the decision, then it may make sense to treat λ as a choice variable, rather than taking it to be exogenous. Smith and Walker [1993] use a model like this to explain departures from individual decision making, in which there is a tradeoff between costly effort of making an optimal choice and the expected utility gain from being careful. This model, or something like it, could be extended by imposing the logit equilibrium conditions. A second way to endogenize λ is to construct specific learning models, which could explicitly model the changes in λ resulting from experience. As history of past play accumulates, then players should be better able to estimate the expected payoffs, at least for those strategies that have been used in the past. This would produce stochastic versions of models that have been looked at in the past, such as fictitious play, cournot dynamics, and linear adjustment processes.

7.5 Games in extensive form

A number of interesting new issues arise in the application of the logit equilibrium to extensive form games. The technical details, as well as a variety of applications to experimental games, are discussed in McKelvey and Palfrey (1995b).

We begin by defining a game in its extensive form in the standard way. Let $I = \{0, 1, \ldots, I\}$ be the set of **players**, with (player "0") called **nature**. The game is represented by:

- a set of **nodes**, with elements $x \in X$
- an initial node, denoted x_0
- a set of terminal nodes $Z \subseteq X$
- an immediate precedence relation P defined on $X \times X$ representing **branches**. If xPx' then x **immediately precedes** x' and x' **immediately follows** x. P is asymmetric and acyclic.
- a partition of the non-terminal nodes, $X - Z$, into $I + 1$ subesets, $X^0, X^1, \ldots, X^I$, with X^0 called **Nature's moves** and X^i called the **moves of player** i.

- for each $x \in X^0$, a probability distribution over its immediate followers.
- for each i a partition of X^i into $k(i)$ information sets, $h_{i1}, h_{i2}, \ldots, h_{ik(i)}$. The set of player i's information sets by H_i.
- a set of available actions, A_{ij}, associated with each information set, and the union of all these actions over all information sets is denoted A_i.
- a payoff function for each player, u^i that assigns a real number to each element of Z.

For each terminal node $z \in Z$ there is a unique path connecting the initial node, which consists of a sequence of moves by one or more players. We will assume throughout that the game has perfect recall. That is, every player knows his past actions and knows everything he knew at any of his previous information sets.

A **behavior strategy** for player $i \in I$ is a function $b_i : H_i \to \mathcal{M}(A_i)$ satisfying $b_i(h_{ij}) \in \mathcal{M}(A_{ij})$ for all i and for all $h_{ij} \in H_i$, where $\mathcal{M}(C)$ indicates the set of all probability measures over the set C. For $h_{ij} \in H_i$ we use the shorthand $b_{ij} = b_i(h_{ij})$. For each $a \in A_{ij}, b_{ij}(a)$ denotes the probability of action a. Let B_i denote the set of all behavior strategies for player i, and $B = \prod_{i \in N} B_i$ be the set of behavior strategy n-tuples. A pure strategy for i is a behavior strategy that assigns a degenerate probability distribution to each $h_{ij} \in H_i$.

The idea of a logit equilibrium in an extensive form game is the same as in the normal form model, except that the logit response probabilities are defined on the *conditional* expected payoffs at a decision node, and a version of sequential rationality is automatically[13] imposed. Given a behavior strategy, b, and any information set h_{ij}, and any action $a_{ijk} \in A_{ij}$ we can define the conditional distribution over the terminal nodes for that action. From this conditional distribution over terminal nodes, we can compute the expected payoff for each $a_{ijk} \in A_{ij}$, conditional on reaching A_{ij}. Denote this by $v(a_{ijk}; b)$. Then the logit model simply requires that the probability of choosing a_{ijk} at information set h_{ij} is given by

$$b_{ij}(a_{ijk}) = \frac{e^{\lambda v(a_{ijk};b)}}{\sum_{a' \in A_{ij}} e^{\lambda v(a';b)}}.$$

A **Logit Equilibrium** in the extensive form, is any b satisfying this equation for all $i, j, a' \in A_{ij}$. We also refer to this as the **Agent Logit Equilibrium**, because it can be justified as a Bayesian equilibrium of the agent model[14] of the extensive form game, in

[13]It is automatically imposed, in the sense that the logit model implies all decision nodes can be reached, and players are responding to their conditional payoffs.

[14]In the agent version of an extensive form game, each information set of a player is interpreted as a different agent of that player, but all agents of the same player share identical preferences over the terminal nodes.

	2	1
2	2,2	2,2
1	0,0	3,1

Table 5.
Normal form of simple ultimatum game.

which the payoff perturbations are with respect to each of the conditional payoffs of each agent at each information set. Thus it is as if the different agents of a player observe different information, which none of the other players, including the other agents of the same player, observe. The Bayesian equilibria of such a game, when the perturbations are distributed according to a double exponential distribution with parameter λ, are equivalent (in terms of path probabilities) to the logit equilibria.

Example: The invariance principle

An interesting feature of the logit equilibrium in extensive forms is that it depends on "inessential" transformations[15] of the extensive form. Thus it predicts different patterns of behavior for strategically equivalent games. A nice example of this is the simple ultimatum game, with $J = 2$. Consider the game being played in two stages. In the first stage, the first mover makes an offer of either $(2, 2)$, or $(3, 1)$. If the equal split is offered, the second mover must accept it. Otherwise, the second mover can either accept or reject. The normal form of this game is given in Table 5 and the extensive form is given in Figure 6. The logit equilibrium correspondence of the normal form [16] of this game is shown in Figure 7a, and Figure 7b below shows the *unique* extensive form logit equilibrium.

Two comparisons are noteworthy. First, the equilibrium correspondence is unique for the sequentially played version of the game and is not unique for the simultaneous version. Second, for all values of λ, the logit equilibrium (both of them, in fact) in the simultaneous version of the game has a higher probability of a fair offer than the sequential version of the game. This occurs because for all values of λ the conditional probability Player 2 will reject an unfair offer is higher in the simultaneous version. The intuition for this is simple. In the simultaneous move version, the conditional expected utility difference between Player 3's two strategies is less than in the sequential version,

[15]An inessential transformation of a game in extensive form produces another game which has the same reduced normal form.

[16]This is also the extensive form logit equilibrium correspondence for the extensive form game in which Player 2 must choose whether or not to reject an unfair offer, *before* observing the offer. For this reason, we refer to this version of the game as the *simultaneous play* version, and the other as the *sequential play* version

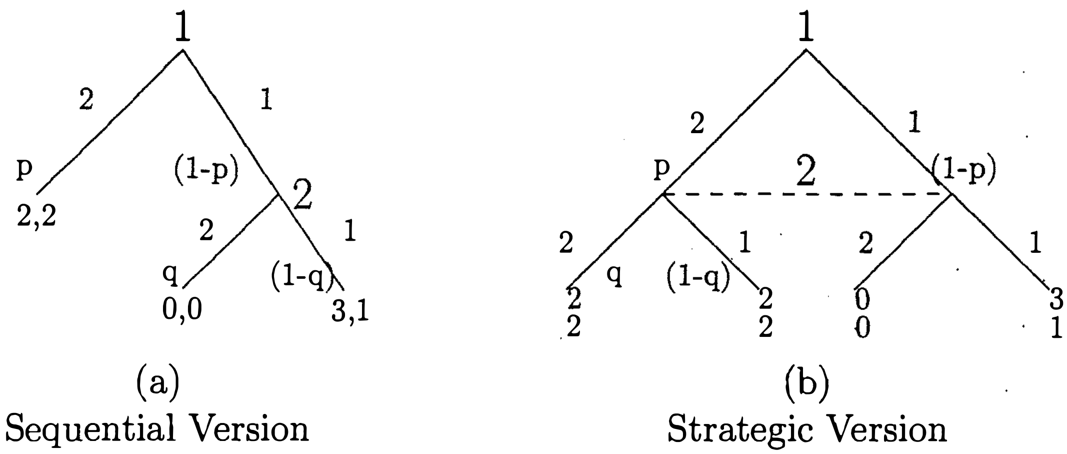


Figure 6: Extensive form of simple ultimatum game:

since Player 2 does not know Player 1's offer at the time he chooses his strategy. Thus the probability of the worse of the two strategies (rejecting unfair offers) is higher in the simultaneous version.

This latter observation is discussed in greater detail in McKelvey and Palfrey (1995b). That paper also analyzes some experiments by Schotter, Weigelt, and Wilson (1994) that provide empirical support for these effects of inessential transformations of the game.

8 Concluding remarks

This paper presented a statistical framework for modelling equilibria in noncooperative games. The framework blends the ideas from Nash equilibrium and Harsanyi's notion of Bayesian equilibrium with standard statistical models of discrete choice. Besides providing new insights into behavior in games, this model provides a potentially useful tool for the econometrics of game theory.

The framework can be easily applied to data analysis as a tool for estimation and testing. This is discussed in detail in McKelvey and Palfrey (1995a, 1995b), which illustrates this with a number of specific applications using experimental data sets. This econometric approach is in the same category of applications to data from experimental games as El-Gamal et al. (1993), El-Gamal and Palfrey (1994a, 1994b) and McKelvey and Palfrey (1992), although those papers did not model the statistical component of behavior as varying monotonically in the expected payoffs.

A number of theoretical insights can be drawn from this model. First, it provides a unique equilibrium selection in generic games. Second, it predicts systematic differences in equilibrium behavior compared with the predictions of Nash equilibrium or some alternative refinements of Nash equilibrium. This was illustrated with discrete versions

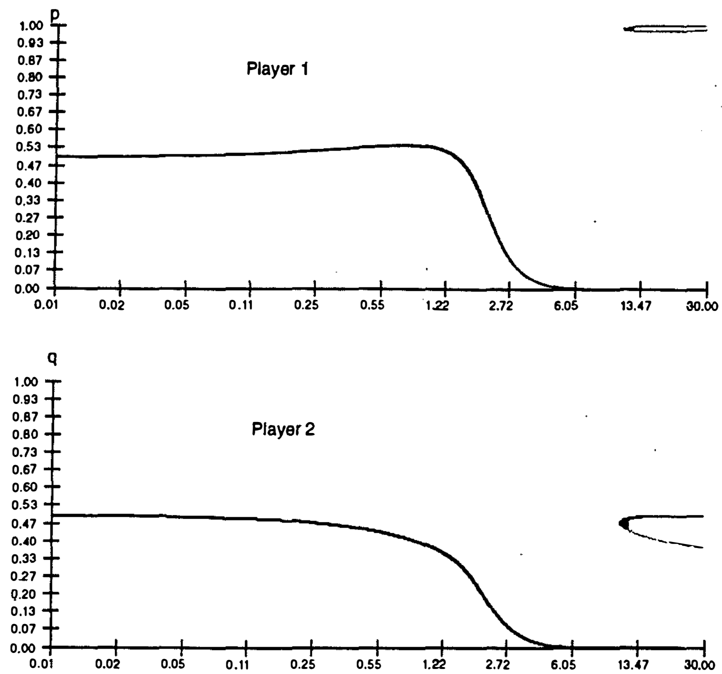


Figure 7a

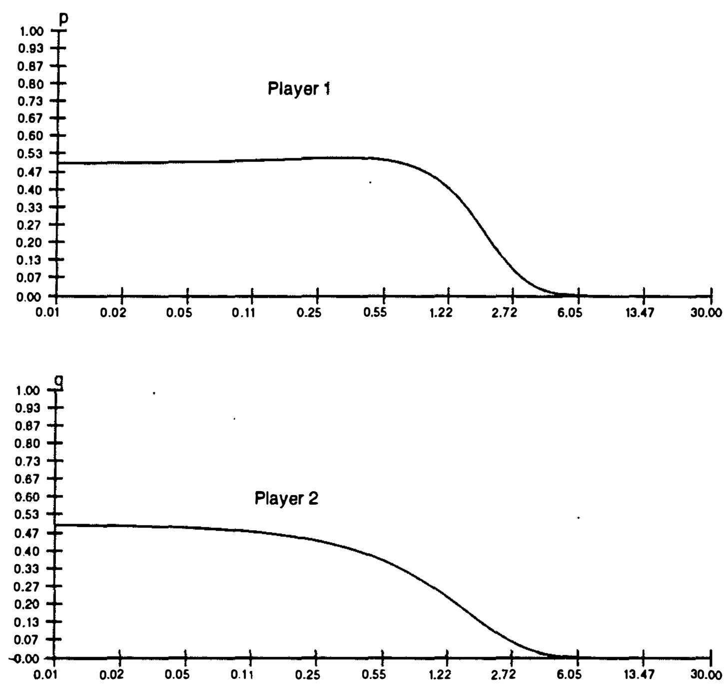


Figure 7b

Figure 7: Logistic QREs for strategic and sequential version of the simple ultimatum game

of the ultimatum game, but applies in other contexts as well. For example, McKelvey and Palfrey (1995b) successfully apply this theory to explain why unintuitive sequential equilibria are played in certain experimental signaling games (Banks et al. 1994, Brandts and Holt 1992, 1993). These systematic differences from Nash equilibrium also imply systematic deviations from the "invariance principle" that behavior should be the same in two different extensive form games as long as they have the same reduced normal form (Kohlberg, E. and J.-F. Metens, 1986).

Finally, there are several directions to extend this framework, some of which were discussed briefly above. These include allowing for heterogeneity, continuous action spaces, learning, and private information.

REFERENCES

Banks, J., C. Camerer, and D. Porter, "Experimental tests of Nash refinements in signaling games," *Games and Economic Behavior*, 6 (1994):1–31.

Beja, A., "Imperfect equilibrium," *Games and Economic Behavior*, 4 (1992):18–36.

Brandts, J., and C. A. Holt, "An experimental test of equilibrium dominance in signaling games," *American Economic Review*, 82 (1992):1350–65.

Brandts, J., and C. A. Holt, "Adjustment patterns and equilibrium selection in experimental signaling games," *International Journal of Game Theory*, 22 (1993):279–302.

Brown, J., and R. Rosenthal, "Testing the minimax hypothesis: A reexamination of O'Neill's game experiment," *Econometrica*, 58 (1990):1065–81.

Chen, H.-I., J. Friedman, J.-F. Thisse, "Boundedly Rational Nash Equilibrium: A Probabilistic Choice Approach," mimeo, University of North Carolina, (1995).

El-Gamal, M., R. D. McKelvey and T. R. Palfrey, "A Bayesian sequential experimental study of learning in games," *Journal of the American Statistical Association*, 88 (1993):428–35.

El-Gamal, M., and T. R. Palfrey, "Vertigo: comparing structural models of imperfect behavior in experimental games," *Games and Economic Behavior*, (1994a) in press.

El-Gamal, M. and T. R. Palfrey, "Economical experiments: Bayesian efficient experimental design," Social Science Working Paper, California Institute of Technology, (1994b), *International Journal of Game Theory*, forthcoming.

Fey, M., R. D. McKelvey and T. R. Palfrey, "Experiments on the constant-sum centipede game," *International Journal of Game Theory*, forthcoming.

Friedman, J., Palfrey, T. R., and J.-F. Thisse, "Random Choice Behavior in Continuum Models," mimeo, University of North Carolina, (1995).

Fudenberg, D. and J. Tirole, *Game Theory*, Cambridge: MIT Press, 1991.

Gale, J., K. Binmore, and L. Samuelson, "Learning to be imperfect: The ultimatum game," SSRI Working Paper No. 9323, University of Wisconsin, (1993).

Güth, W and R. Tietz, "Ultimatum bargaining for a shrinking cake – An experimental analysis," in R. Tietz, W. Albers, and R. Selten, (eds.) *Bounded Rational Behavior in Experimental Games and Markets*, Berlin: Springer Verlag, (1988).

Güth, W and R. Tietz, "Ultimatum Bargaining Behavior: A Survey and Comparison of Experimental Results," *Journal of Economic Psychology*, 11 (1990):417–49.

Harsanyi, J., "Games with randomly disturbed payoffs," *International Journal of Game Theory*, 2 (1973):1–23.

Harsanyi, J. and R. Selten, *A general theory of equilibrium selection in games,* Cambridge: Massachusetts Institute of Technology Press, 1988.

Kohlberg, E. and J.-F. Mertens, "On the strategic stability of equilibria," *Econometrica*, 54 (1986):1003–37.

Luce, D., *Individual Choice Behavior*, New York: Wesley, 1959.

Luce, D., and P. Suppes, "Preference, utility and subjective probability," in R. D. Luce, B. Bush, and E. Galanter (eds.) *Handbook of Mathematical Psychology, vol. III*, New York: Wiley, (1965).

Ma, C.A., and M. Manove, "Bargaining with deadlines and imperfect player control," *Econometrica*, 61 (1993):1313–39.

McFadden, D. "Conditional logit analysis of qualitative choice behavior," in P. Zarembka (ed.) *Frontiers of Econometrics*, New York: Academic Press, 1973.

McFadden, D. "Quantal choice analysis: a survey," *Annals of Economic and Social Measurement*, 5 (1976):363–90.

McKelvey, R. D. and T. R. Palfrey, "An experimental study of the centipede game," *Econometrica*, 60 (1992):803–36.

McKelvey, R. D. and T. R. Palfrey, "Quantal Response Equilibria for Normal Form Games," *Games and Economic Behavior*, 10 (1995a):6–38.

McKelvey, R. D. and T. R. Palfrey, "Quantal Response Equilibria for Extensive Form Games," mimeo, California Institute of Technology, (1995b).

Ochs, J., "Games with unique mixed strategy equilibria: An experimental study," *Games and Economic Behavior*, 10 (1995):202–217.

O'Neill, B., "Nonmetric test of the minimax theory of two person zerosum games," *Proceedings of the National Academy of Science, USA*, 84 (1987):2106–2109.

Palfrey, T. R. and H. Rosenthal, "Testing game-theoretic models of free riding: new evidence of probability bias and learning," in T. Palfrey (ed.) *Laboratory Research in Political Economy*, Ann Arbor: University of Michigan Press, (1991).

Rosenthal, R., "A bounded-rationality approach to the study of noncooperative games," *International Journal of Game Theory*, 18 (1989):273–92.

Schmidt, D., "Reputation building by error-prone agents," mimeo (1992), California Institute of Technology.

Schotter, A., K. Weigelt, and C. Wilson, "A Laboratory Investigation of Multiperson Rationality and Presentation Effects," *Games and Economic Behavior*, 6 (1994):445–68.

Smith, V. and J. Walker, "Monetary rewards and decision cost in experimental economics," *Economic Inquiry*, 31 (1993):245–61.

Stahl, D., and P. Wilson, "On players' models of other players - a new theory and experimental evidence," *Games and Economic Behavior*, 10 (1995):218–246.

Thaler, R., "The Ultimatum Game," *Journal of Economic Perspectives*, 2 (1988):195–206.

Thurstone, L., "A law of comparative judgement," *Psychological Review*, 34 (1927):273-286.

Van Damme, E., *Stability and Perfection of Nash Equilibria*, Berlin: Springer-Verlag, 1987.

Zauner, K., "Bubbles, speculations and a reconsideration of the centipede game experiment," mimeo, UC San Diego, (1993) (revised as "A Reconsideration of the Centipede Game Experiments," 1994).